HEAT EQUATION APPROACH TO INDEX THEOREMS ON ODD DIMENSIONAL MANIFOLDS

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ABSTRACT. D.Freed has formulated and proved an index theorem on odd dimensional spin manifolds with boundary. The proof is based on analysis by Calderon and Seeley. In this note we are going to give a proof of this theorem using the heat kernels methods for boundary conditions of Dirichlet and Von Neumann type. Moreover we consider also the Atiyah-Patodi-Singer spectral boundary condition which is not considered in Freed's paper. As a direct consequence of the method, we will obtain some information about isospectral invariants of the boundary conditions. This proof does not uses the cobordism invariance of index and are easily generalized to family case.

1. INTRODUCTION

Dirac operators on compact odd dimensional manifolds, without boundary, are formally self adjoint, so their index will be zero and have no interest. However, if the underlying manifold has boundary, by puttin suitable boundary conditions, one can obtain non self adjoint operators with non zero index. In [Fre], D. Freed, inspired by the work of physicians has formulated a such index theorem and proved it by means of symbol calculus of elliptic boundary problems. In this paper we give a heat equation proof of this theorem and consider, in addition, the case of spectral boundary conditions. To compute the contribution of APS condition, we use the perfect symmetry between positive and negative part of the spectrum of Dirac operators in even dimension rather than using calculus which are established to handle this problem in general (see, eg. [MP] and [BC]). In section 2 we state the theorem and prove it for one Dirac operator. At the end of this section we will give a necessary condition for isospectrality of these boundary problems with exchanged boundary conditions on different connected components of boundary. In section 3 we will formulate and prove the index theorem for a family of Dirac operators by studying the Chern character of a superconnection adopted to the family of Dirac operators. Although our notation refers to the Dirac operator acting on standard complex spinor fields, the method can be applied directly to the case of Dirac type operators acting on twisted Dirac operator or Dirac operators acting on Clifford bundles.

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2. INDEX THEOREM FOR A DIRAC TYPE OPERATOR

Let M be a compact spin manifold of dimension $2n + 1$ with a Riemannian metric taking the product form $d^2u + g$ in a collar neighborhood $U = [0, 1^+) \times \partial M$ of the boundary. Here 1^+ may be any real number greater than 1. In the sequel we denote the coordinate of the interval $[0, 1^+)$ by u and v and those of ∂M by y and z. A typical point of M will be denoted by x. We fix a spin structure on M giving rise to the complex spin vector bundle S on M. The orientation and spin structure on M induce an orientation and a spin structure on ∂M. So the restriction of spin bundle S to the collar neighborhood of the boundary splits into the positive and negative parts

$$S_{|U} \simeq S^+ \oplus S^-$$

where $S^+ \oplus S^-$ is the spin bundle of ∂M lifted to the collar neighborhood U in the obvious manner. According to this splitting, each spinor field $\phi \in C^\infty(M, S)$ has a decomposition $\phi_{|U} = \phi^+ \oplus \phi^-$, where $\phi^\pm \in C^\infty(\partial M, S^\pm)$. With respect to this decomposition, the Dirac operator D has the following form where A denote the Dirac operator of ∂M (see, eg. [BBW, chapter 9])

$$\begin{pmatrix} i\partial_u & iA^- \\ -iA^+ & -i\partial_u \end{pmatrix}.$$

Let $\partial M = \sqcup_i N_i$ where N_i's are disjoint union of connected components of boundary. For each i, choose ϵ_i be 0, $+$ or $-$ arbitrarily and fix them. For $\epsilon_i = \pm$ let P^{ϵ_i} denote the following local boundary condition

$$(2.1) \qquad\qquad (\phi_{|N_i})^{\epsilon_i} = 0.$$

Corresponding to $\epsilon_i = 0$, P^0 denotes the Atiyah-Patodi-Singer(APS) boundary condition. We recall briefly its definition. The Dirac operator A_i of each connected component N_i of ∂M has a discrete resolution $\{\phi_\lambda, \lambda\}$. Since N_i is even dimensional, there is a symmetry betweed positive and negative part of spectrum. This symmetry is given by the following unitary isomorphism $\mathcal{U}$ between eigenspaces with opposite eigenvalues

$$(2.2) \qquad\qquad \mathcal{U}(\phi_\lambda^+ + \phi_\lambda^-) = \phi_\lambda^+ - \phi_\lambda^- \; ; \; \phi_\lambda = \phi_\lambda^+ + \phi_\lambda^-.$$

This symmetry is responsible for the eta-invariant of N vanishes. Let $P^0 \in \mathrm{End}(\mathrm{L}^2(N_i.S))$ be the orthogonal projection on the subspace generated by ϕ_λ's with $\lambda \geq 0$. This projection defines the APS boundary condition. By P^ϵ we denote the boundary condition which equals P^{ϵ_i} on N_i.

Let ϕ and ψ be smooth spinor field on M, then the following formula holds

$$(2.3) \qquad\qquad < D\,\phi, \psi >_{L^2} - < \phi, D\,\psi >_{L^2} = -\int_{\partial M} < \mathrm{cl}(\frac{\partial}{\partial u})\phi, \psi > \, dy.$$

So it turns out that boundary problems (D, P^+) and (D, P^-) are adjoint, one for other. Moreover the adjoint problem for (D, P^0) is $(D, Id - P^0)$. We denote the formal adjoint of the boundary problem $D := (D, P^\epsilon)$ by $D^* := (D, P^{\bar\epsilon})$. These boundary problems are elliptic so they have finite dimensional kernels consisting of smooth spinor fields. Therefore one can define the index of this problem by

$$(2.4) \qquad\qquad \mathrm{ind}(D, P^\epsilon) = \dim \ker D^*D - \dim \ker DD^*.$$

Theorem 1 (See theorem B of [Fre]). *Denote by A_i the Dirac operator of the even dimensional manifold N_i. The following formula holds*

$$\mathrm{ind}(D, P^\epsilon) = \frac{1}{2} \sum_{\epsilon_i=-} \mathrm{ind}\, A_i - \frac{1}{2} \sum_{\epsilon_i=+} \mathrm{ind}\, A_i - \frac{1}{2} \sum_{\epsilon_i=0} \dim \ker A_i.$$

To prove this theorem, we consider these boundary problems on half cylinder $\mathbb{R}^{\geq 0} \times N$ with product spin structure and product Riemannain metric. Here N may denote each one of N_i's and A denotes its Dirac operator.

2.1. **Index density of local boundary conditions on half cylinder.** We consider the boundary condition P^+; the case P^- can be studied in the same manner. On this half cylinder the Dirac operator D, acting on compactly supported spinor fields subjected to condition P^+, takes the following form

$$(2.5) \qquad\qquad \begin{pmatrix} i\partial_u & iA^- \\ -iA^+ & -i\partial_u \end{pmatrix}.$$

We denote the adjoint problem by D^*. The Laplacian operators D^*D and DD^*, acting on compactly supported spinor fields $\phi(u,y)$, are non negative operators taking the form $-\partial_u^2 + A^2$. The induced boundary condition for D^*D is

$$\phi^+(0,y) = 0 \;\; \text{and} \;\; (\partial_u \phi^- + A^+\phi^+)_{|u=0} = 0.$$

Since A is a tangential operator, these conditions reduce to Dirichlet condition for ϕ^+ and Von Neumann condition for ϕ^-

$$(2.6) \qquad\qquad \phi^+(0,y) = 0$$

$$(2.7) \qquad\qquad \frac{\partial \phi^-}{\partial u}(0,y) = 0.$$

Concerning the operator DD^*, a similar argument shows that the boundary conditions take the following forms

$$(2.8) \qquad\qquad \frac{\partial}{\partial u}\psi^+(0,y) = 0 \;\; \text{and} \;\; \psi^-(0,y) = 0.$$

Consider, for $t > 0$, the heat operators e^{-tD^*D}. The kernel of this operator, $\bar{K}_1(t, u, v, y, z)$, with respect to the boundary condition (2.6) has the following explicit form, cf. [MS]

$$(2.9) \qquad \frac{1}{\sqrt{4\pi t}}\{\exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t})\}e^{-tA^-A^+}(t, y, z),$$

while the heat kernel for boundary condition (2.7) is

$$(2.10) \qquad \frac{1}{\sqrt{4\pi t}}\{\exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t})\}e^{-tA^+A^-}(t, y, z).$$

So the trace density of heat operator e^{-tD^*D}, as a function of $t > 0$ and $u \geq 0$ is

$$\bar{K}_1(t, u) := \int_N \mathrm{tr}_y \bar{K}_1(t, u, v, y, y)\, dy$$

$$(2.11) \qquad = \frac{\mathrm{Tr}\, e^{-tA^-A^+}}{\sqrt{4\pi t}}\{1 - \exp(\frac{-u^2}{t})\} + \frac{\mathrm{Tr}\, e^{-tA^+A^-}}{\sqrt{4\pi t}}\{1 + \exp(\frac{-u^2}{t})\}.$$

Denote by $\bar{K}_2(t, u, v, y, z)$ the kernel of heat operator e^{-tDD^*} with boundary conditions (2.8). Above discussion gives the following expression for trace density of this heat operator

$$(2.12) \qquad \bar{K}_2(t, u) = \frac{\mathrm{Tr}\, e^{-tA^-A^+}}{\sqrt{4\pi t}}\{1 + \exp(\frac{-u^2}{t})\} + \frac{\mathrm{Tr}\, e^{-tA^+A^-}}{\sqrt{4\pi t}}\{1 - \exp(\frac{-u^2}{t})\}.$$

So we obtain the following formula for trace density $\bar{K}_+(t, u) = \bar{K}_1(t, u) - \bar{K}_2(t, u)$ of the operator $e^{-tD^*D} - e^{-tDD^*}$ with boundary condition P^+

$$\bar{K}_+(t, u) = \frac{e^{-\frac{u^2}{t}}}{\sqrt{\pi t}}\{Tr\, e^{-tA^+A^-} - Tr\, e^{-tA^-A^+}\}.$$

Integrating with respect to $u \in \mathbb{R}^{\geq 0}$ we get

$$(2.13) \qquad \int_0^\infty \bar{K}_+(t, u)\, du = -\frac{1}{2}\mathrm{ind}A.$$

Here we have used the McKean-Singer formula $\mathrm{ind}A = \mathrm{Tr}\, e^{-tA^-A^+} - \mathrm{Tr}\, e^{-tA^+A^-}$. Similarly, if $\bar{K}_-(t, u)$ denote the trace density of operator $e^{-tD^*D} - e^{-tDD^*}$ with boundary condition P^-, then

$$\int_0^\infty \bar{K}_-(t, u)\, du = \frac{1}{2}\mathrm{ind}A.$$

If instead of integration on $[0, \infty)$ in above we integrate on the finite interval $[0, \frac{1}{2}]$, the difference, would be of exponential decay when t goes toward 0. In fact we have the following relation

$$(2.14) \qquad \int_0^{\frac{1}{2}} \bar{K}_\pm(t, u)\, du = \mp\frac{1}{2}\mathrm{ind}A + \mathrm{O}(e^{-\frac{1}{4t}})$$

Following relation is a result of above discussion and we write down it here for future reference

$$(2.15) \qquad \bar{K}_\pm(t, u) \sim 0 \quad \text{exponentially at } t = 0 \text{ for } u \neq 0.$$

Remark 1. *If $y \neq z$, then it is well known that the heat kernel $e^{-tA^-A^+}(t, y, z)$ and $e^{-tA^+A^-}(t, y, z)$, as well as thir derivatives with respect to y, are exponentially small at $t = 0$. If $y = z$ and $u \neq v$ the expression given in formulas (2.9) and (2.10) have this property. In this case the differentiation may be taken with respect to t.*

2.2. **Index density of APS boundary problem on half cylinder.** Now let $\epsilon_i = 0$ and consider the half cylinder $\mathbb{R}^{\geq 0} \times N$ with boundary problems $D = (D, P^0)$ and $D^* = (D, Id - P^0)$. The induced boundary condition for D^*D is

$$P^0(\phi) = 0 \quad \text{and} \quad (Id - P^0)D\phi = 0.$$

Exchanging the role of P and $(Id - P)$ we obtain the adjoint induced boundary condition Q^* for D^*D. Denote by E_λ the product vector bundle on $\mathbb{R}^{\geq 0}$ whose fibers are the λ-eigenspace of A. Operators

D^*D and DD^* takes the form $-\partial_u^2 + \lambda^2$ on sections of E_λ. Put $A_\lambda := A_{|E_\lambda}$. Boundary conditions on section $\phi(u,y)$ of E_λ, for D^*D, read

(2.16) $$\phi(0,y) = 0 \qquad\qquad \text{for } \lambda \geq 0 ;$$

(2.17) $$(\frac{\partial}{\partial u} + \lambda)_{|u=0}\, \phi(u,y) = 0 \qquad\qquad \text{for } \lambda < 0.$$

Following boundary conditions must be considered for DD^*.

(2.18) $$\psi(0,y) = 0 \qquad\qquad \text{for } \lambda < 0 ;$$

(2.19) $$(\frac{\partial}{\partial u} + \lambda)_{|u=0}\, \psi(u,y) = 0 \qquad\qquad \text{for } \lambda \geq 0.$$

Let K_1^λ and K_2^λ denote, respectively, the heat kernels of D^*D and DD^*. Let $\bar{K}(t,u)$ denote the supertrace density [1] $\mathrm{Tr_N}\, e^{-tD^*D}(t,u) - \mathrm{Tr_N}\, e^{-tDD^*}(t,u)$. We are interested in the following quantity which is in fact the index of D, acting on spinor fields supported in $[0,1/2] \times N$ and subjected to APS condition at $\{0\} \times N$

(2.20) $$\int_0^{\frac{1}{2}} \bar{K}(t,u) = \sum_\lambda \int_0^{\frac{1}{2}} K_1^\lambda(t,u) - \int_0^{\frac{1}{2}} K_2^{-\lambda}(t,u).$$

We recall operator $\mathcal{U}$ provides, for $\lambda \neq 0$, a unitary isomorphism

$$\mathcal{U}_\lambda := \mathcal{U} : C^\infty(\mathbb{R}^{\geq 0}, E_\lambda) \to C^\infty(\mathbb{R}^{\geq 0}, E_{-\lambda})$$

One has $-A_{-\lambda} = \mathcal{U}_\lambda A_\lambda \mathcal{U}_\lambda^{-1}$, so $DD^*_{|E_{-\lambda}} = \mathcal{U}_\lambda D^*D_{|E_\lambda}\mathcal{U}_\lambda^{-1}$. Moreover $\mathcal{U}_\lambda$ exchanges the boundary conditions (2.16) and (2.18). So in expression (2.20), the terms indexed by $\lambda > 0$ cancel themselves. The trace density of fundamental solution of $\partial_t - \partial_u^2 + \lambda^2 = 0$ with boundary condition (2.17) is given by following expression, cf. [APS, relation 2.17]

$$K_1^\lambda(t,u) = \left(\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\{1 + \exp(\frac{-u^2}{4t})\} + \lambda e^{-2\lambda u}\mathrm{erfc}\{\frac{u}{\sqrt{t}} - \lambda\sqrt{t}\} \right).\dim E_\lambda,\ \lambda < 0.$$

The fundamental solution with respect to boundary condition (2.19) is

$$K_2^{-\lambda}(t,u) = \left(\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\{1 + \exp(\frac{-u^2}{4t})\} - \lambda e^{2\lambda u}\mathrm{erfc}\{\frac{u}{\sqrt{t}} + \lambda\sqrt{t}\} \right).\dim E_{-\lambda},\ \lambda < 0.$$

The error function is defined by the following formula

$$\mathrm{erfc(x)} = \frac{2}{\sqrt{\pi}} \int_{\mathrm{x}}^{\infty} e^{-\mathrm{s}^2}\, d\mathrm{s}.$$

Subtracting above expressions, we get the following relations at $t = 0$ for $\lambda < 0$

$$K_1^\lambda(t,u) - K_2^{-\lambda}(t,u) \sim 0 \quad \text{exponentially for } u \neq 0,$$
$$K_1^\lambda(t,0) - K_2^{-\lambda}(t,0) \sim \lambda + o(1).$$

So

$$\int_0^{\frac{1}{2}} K_1^\lambda(t,u) - K_2^{-\lambda}(t,u)\, du \sim 0 \quad \text{exponentially at } t = 0.$$

Therefore, when t goes zero, the only terms with probably nonzero contribution in relation (2.20) are indexed by $\lambda = 0$. For $\lambda = 0$, the boundary conditions (2.16) and (2.19) are respectively the Dirichlet and Von Neumann boundary conditions. Regarding expression (2.9) and (2.10) we have

(2.21) $$K_1^0(t,u) - K_2^0(t,u) = -\frac{e^{\frac{-u^2}{t}}}{\sqrt{\pi t}}\dim \ker A.$$

Hence

$$\int_0^{\frac{1}{2}} (K_0^1(t,u) - K_0^2(t,u))\, du \sim -\frac{1}{2}\dim \ker A \quad \text{exponentially at } t = 0.$$

We summarize above discussion in following proposition

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[1] Here Tr and STr denote the L^2 trace while tr and str denote the pointwise trace of finite dimensions vector space

Proposition 2. *Denote the heat kernel of boundary problems D^*D and DD^*, respectively, by $\bar{K}_1$ and $\bar{K}_2$. With above notation, put $E^+ := \oplus_{\lambda > 0} E_\lambda$ and $E^- := \oplus_{\lambda < 0} E_\lambda$. $\mathcal{U}$ provides a natural unitary isomorphism between these spaces and satisfies*

$$\bar{K}_{2|E^-} = \mathcal{U} \bar{K}_{1|E^+} \mathcal{U}^{-1}$$
$$\bar{K}_{2|E^+} \sim \mathcal{U} \bar{K}_{1|E^-} \mathcal{U}^{-1} + o(1) \quad \text{exponentially at } t = 0.$$

Moreover, with $\bar{K}(t,u) := \bar{K}_1(t,u) - \bar{K}_2(t,u)$ we have

(2.22)
$$\int_0^{\frac{1}{2}} \bar{K}(t,u) \, du \sim -\frac{1}{2} \dim \ker A \quad \text{exponentially at } t = 0.$$

Remark 2. *For later use we rewrite the following result of above discussion*

(2.23)
$$\bar{K}(t,u) \sim 0 \quad \text{exponentially for } u \neq 0.$$

2.3. Index theorem for one Dirac operator.

The heat equation proof of the theorem 1 is based on the Mckean-Singer formula. Let D denote the boundary problem (D, P^ϵ) and D^* denote its adjoint problem $(D, P^{\bar{\epsilon}})$. These boundary problems are elliptic, so the elements of $\ker(D, P^\epsilon)$ are smooth spinor fields which are exactly those elements of $\ker D^*D$ satisfying induced boundary conditions. Similar remark applies to $\ker(D^*, P^{\bar{\epsilon}})$ and $\ker DD^*$. Hence using the spectral resolution of D^*D and DD^* one has the following Mckean-Singer type equality

$$\text{ind}(D, P^\epsilon) = \text{Tr}\, e^{-tD^*D} - \text{Tr}\, e^{-tDD^*} \; ; \; t > 0.$$

Denote by $\bar{K}_{1\epsilon}$ and $\bar{K}_{2\bar{\epsilon}}$, respectively, the fundamental solutions of the cylindrical heat operators e^{-tD^*D} and e^{-tDD^*}. Let K_ϵ and $K_{\bar{\epsilon}}$ be, respectively, the fundamental solution of e^{-tD^*D} and e^{-tDD^*}. We are going to give asymptotic expressions, at $t = 0$, for these fundamental solutions in term of $\bar{K}_{1\epsilon}$, $\bar{K}_{2\bar{\epsilon}}$ and of the fundamental solution $\tilde{K}$ of e^{-tDD} on the double of M, i.e. $M \sqcup_{\partial M} M^-$. For this purpose, following [APS, Page 54], let $\rho(a,b)$ be a smooth increasing function on $\mathbb{R}^{\geq 0}$ such that

$$\rho(u) = 0 \text{ for } u \leq a \; ; \; \rho(u) = 1 \text{ for } u \geq b.$$

Collar neighborhoods of connected component of boundary are assumed being parameterized by $u \in [0, 1^+]$, so the following functions may be considered as smooth function on M with constant extension into M.

$$f_2 = \rho(\frac{1}{4}, \frac{1}{2}), \qquad\qquad\qquad g_2 = \rho(\frac{1}{2}, \frac{3}{4})$$
$$f_1 = 1 - \rho(\frac{3}{4}, 1), \qquad\qquad\qquad g_1 = 1 - g_2$$

Put

(2.24)
$$\mathcal{K}_\epsilon = f_1 \bar{K}_{1\epsilon} g_1 + f_2 \tilde{K} g_2 \quad \text{and} \quad \mathcal{K}_{\bar{\epsilon}} = f_1 \bar{K}_{2\bar{\epsilon}} g_1 + f_2 \tilde{K} g_2.$$

Since $f_i = 1$ on the support of g_i, one conclude that $\mathcal{K}_\epsilon$, as an operator on $C^\infty(M, S)$, goes toward Id when $t \to 0$. Moreover the remark 1 show that $(\frac{\partial}{\partial t} + D^*D)\mathcal{K}_\epsilon$ is exponentially small, out of diagonal, when $t \to 0$. These two condition are sufficient for using $\mathcal{K}_\epsilon$ as the initial step in construction of heat kernel using the Levi's sum, cf. [MS]. As a consequence, the difference between heat kernel K_ϵ and $\mathcal{K}_\epsilon$ is exponentially small when t goes toward 0. This argument applies also to $K_{\bar{\epsilon}}$ and $\mathcal{K}_{\bar{\epsilon}}$. Therefore we have

$$\text{ind}(D, P^\epsilon) = \int_{diag(M)} \{tr\, K_\epsilon(t,x,x) - tr\, K_{\bar{\epsilon}}(t,x,x)\}$$
$$= \lim_{t\to 0} \int_{diag(M)} \{tr\, \mathcal{K}_\epsilon(t,x,x) - tr\, \mathcal{K}_{\bar{\epsilon}}(t,x,x)\}$$
$$= \lim_{t\to 0} \sum_i \int_0^{\frac{1}{2}} \bar{K}_{1\epsilon}(t,u) - \bar{K}_{2\bar{\epsilon}}(t,u)$$
$$= \lim_{t\to 0} \sum_i \int_0^{\frac{1}{2}} \bar{K}_\epsilon(t,u) \, du.$$

To deduce the last equality, we have used the fact that the contribution of the trace of the heat operator on double $M \sqcup M^-$ is the same in the expressions $tr \, \mathcal{K}_\epsilon$ and $tr \, \mathcal{K}_{\bar{\epsilon}}$. We have use also $f_1(u) = g_1(u) = 1$ for $0 \leq u \leq \frac{1}{2}$. Now relations (2.13), (2.14) and (2.22) imply the desired formula

$$\text{ind}(D, P^\epsilon) = \frac{1}{2}\sum_{\epsilon=-}\text{indA}_i - \frac{1}{2}\sum_{\epsilon=+}\text{indA}_i - \frac{1}{2}\dim\ker(A_i).$$

2.4. **Isospectrality problem.** In this subsection we denote by Q^ϵ the second degree boundary condition associated to P^ϵ. As we have mentioned before, for $t > 0$, the heat operator e^{-tD^*D} is a compact smoothing self adjoint operator, so the boundary condition problem (D^*D, Q^ϵ) has a resolution $\{(\phi_\lambda, \lambda)\}$. Let ϵ' be an another set of boundary conditions. For simplicity let ϵ_i and ϵ'_i be $+$ or $-$. Let

$$e^{-tA_i^- A_i^+} \sim t^{-n}\sum_{k=0}^\infty a_{ik}^+ t^k \ ,$$

$$e^{-tA_i^+ A_i^-} \sim t^{-n}\sum_{k=0}^\infty a_{ik}^- t^k$$

be the asymptotic expansions of heat operators on N_i. It follows from Mckean-Singer formula that

(2.25) $$a_{ik}^+ = a_{ik}^- \ \text{ for } k \neq n \ \text{ and } a_{in}^+ = a_{in}^- + \text{indA}_i.$$

If the boundary problems (D^*D, Q^ϵ) and $(D^*D, Q^{\epsilon'})$ have the same spectrum then for $t > 0$ the trace difference $Tr(e^{-tD^*D}, Q^\epsilon) - Tr(e^{-tD^*D}, Q^{\epsilon'})$ vanishes. We are interested to study the asymptotic behavior, at $t = 0$, of this trace difference to get some spectral invariant of these kind of boundary problems. Relations (2.24) and discussion following them show that this trace difference is asymptotic to the following expression at $t = 0$

$$\int_M \mathcal{K}_\epsilon(t, x, x) - \int_M \mathcal{K}_{\epsilon'}(t, x, x) = \int_0^{1/2}\int_{\partial M} \bar{K}_{1\epsilon}(t, y, u) - \bar{K}_{1\epsilon'}(t, y, u)\, dy\, du.$$

On the other hand, relations (2.11) and (2.12) can be used to deduce that the last expression is in its turn asymptotic to

$$(\frac{1}{(4\pi t)})^{\frac{n+1}{2}}\sum_{k,i}\{\sum_{\epsilon_i=+}t^k a_{ik}^- + \sum_{\epsilon_i=-}t^k a_{ik}^+ - \sum_{\epsilon'_i=+}t^k a_{ik}^- - \sum_{\epsilon'_i=-}t^k a_{ik}^+\}.$$

In view of relations (2.25) this last expression simplifies to the following one

$$\sum_{\epsilon_i=-,\epsilon'_i=+}\text{indA}_i - \sum_{\epsilon_i=+,\epsilon'_i=-}\text{indA}_i.$$

So, using the cobordism invariance of index $\sum_{\epsilon=\pm}\text{indA}_i = 0$, we get the following theorem

Theorem 3. *With notation of theorem 1, if two boundary problem (D^*D, Q^ϵ) and $(D^*D, Q^{\epsilon'})$ are isospectral then*

$$\sum_{\epsilon_i=-,\epsilon'_i=+}\text{indA}_i = \sum_{\epsilon_i=+,\epsilon'_i=-}\text{indA}_i = 0.$$

Moreover this condition is the only one which can be deduced from heat equation asymptotic formulas.

Remark 3. *Comparing this result with the case of scalar Laplacian should be interesting. Let $(M, \partial M)$ be a smooth manifold with $\partial M = N_0 \sqcup N_1$. Let $\triangle$ be the scalar Laplacian acting on smooth functions which are subjected to Dirichlet condition on N_0 and Von Neumann condition on N_1. It follows directly from relations (2.9) and (2.10) that all heat equation invariants[2] of $A_{|N_1}$ are spectral invariants for $\triangle$ with this mixed boundary conditions. In particular $Vol(N_1)$, so $Vol(N_0)$, are included in the spectral invariants. Thought this may seem in contrast with above theorem, notice that our boundary conditions are at the same time of Dirichlet and Von Neumann type.*

Remark 4. *Let P^+ denote $\epsilon_i = +$ for all i and define P^- similarly. Using the cobordism invariance of index, the condition of above theorem is satisfied for these pure boundary conditions. This leads to the following interesting question: Does there exist a spin manifold with boundary such that (D^*D, Q^+) and (D^*D, Q^-) be isospectrum.*

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[2]This means the invariant coming from asymptotic expansion, at $t = 0$, of $Tr\, e^{-t\triangle}$

3. Index theorem for families

Theorem 1 can be generalized to include families of Dirac type operators. At first we recall geometric setting for family index theorems.

Let $M \hookrightarrow F \rightarrow B$ be a fibration of odd dimensional compact spin manifolds, with boundary, over a compact smooth manfold B. The boundaries of fibers form an another fibration $\partial M \hookrightarrow F' \rightarrow B$. This boundary fibration has a fibred collar neighborhood U of the form $[0, 1^+) \times F' \subset F$ which restricts to a collar neighborhood in each fiber. We denote by π the projection on the second factor. Assume that the fibration F is endowed with a smooth fiberwise Riemannian metric which is of product form $d^2 u + g$ in the collar neighborhood U. Here g is a fiberwise metric on the boundary fibration F'. We assume also that the fiberwise spin structure is of product form in U. By these assumption, a typical fiber M of the fibration F satisfies conditions described in previous section. So in the sequel we will use freely the previous notation in this family context. Let D and A denote, respectively, the fiberwise family of Dirac operators associated to fibrations F and F'. Put $F' = \sqcup_i N_i$ where each N_i is a connected fibration over B whose fibers are even dimensional compact spin manifolds. For each i, let ϵ_i be 0, $+$ or $-$ arbitrarily and fix it. The family (D, P^ϵ) of boundary problems determines an analytic index $[\text{ind}(D, P^\epsilon)]$ in $K_0(B)$. In other hand, the boundary family A determines, in its turn, a class $[\text{ind}A]$ in $K_0(B)$(see [AS]). The equality of these classes is announced in [Fre]. Here we prove the following theorem using the heat equation tools applied to superconnections.

Theorem 4. *The following equality holds in $H^*_{dr}(B)$*

$$\text{Ch}[\text{ind}(D, P^\epsilon)] = \frac{1}{2} \sum_{\epsilon_i = -} \text{Ch}[\text{ind}A_i] - \frac{1}{2} \sum_{\epsilon_i = +} \text{Ch}[\text{ind}A_i] - \frac{1}{2} \sum_{\epsilon_i = 0} \text{Ch}(\ker A_i).$$

For being able to use the language of the theory of super graded differential modules, we consider the direct sum $S \oplus S$ as a smooth family of super spin bundle such that the first and the second summand are respectively the even and the odd part. The Clifford action of a vertical tangent vector $v \in TM_b$ on $S_b \oplus S_b$ is given by the following grading reversing matrix

$$\begin{pmatrix} 0 & \text{cl}(v) \\ \text{cl}(v) & 0 \end{pmatrix}.$$

With this Clifford action, we obtain the following families of grading reversing Dirac operators on F and on F'

$$\mathsf{D} := \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \; ; \; \mathsf{A} := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

Here D denotes the boundary problem (D, P^ϵ) while D^* denote $(D, P^{\bar{\epsilon}})$. These operators act on vertical spinor fields $\phi \oplus \psi$ satisfying boundary conditions $P^\epsilon(\phi) = 0$ and $P^{\bar{\epsilon}}(\psi) = 0$. For each $b \in B$, D_b is a vertical self adjoint odd differential operator. The induced second degree boundary problem is D^2 with boundary conditions (2.6), (2.7) and (2.8). In below we deal with the infinite dimensional bundle $\mathcal{E}$ of vertical spinor fields. A typical fiber $\mathcal{E}_b$ of this bundle is $C^\infty(M_b, S)$. Bundle $\mathcal{E} \oplus \mathcal{E}$ is $\mathbb{Z}_2$-grading in obvious way. To prove theorem 4 we need a connection ∇ on sections of $\Lambda^* B \otimes \mathcal{E}$.

Remark 5. *In following two subsections, we will study the case of a fibration with half cylindrical fibers endowed with some more special connections. Using a partition of unity on half line $\mathbb{R}^{\geq 0}$ we assume that the connection ∇ coincides with theses special connections in collar neighborhood $[0, 1^+) \times F'$.*

We need to extend the action of D and A on sections $\omega \otimes \xi$ of $\Lambda^* B \otimes \mathcal{E}$. The extension are given by following relations

$$D(\omega \otimes \xi) = (-1)^{\deg \omega} \omega \otimes D\xi \; ; \; A(\omega \otimes \xi) = (-1)^{\deg \omega} \omega \otimes A\xi.$$

Now $\nabla \oplus \nabla$, D and A are graded differential operators acting on smooth sections of graded bundle $\Lambda^* B \otimes \mathcal{E} \oplus \Lambda^* B \otimes \mathcal{E}$. In bellow we shall study the superconnections $\mathbb{B} = \mathsf{D} + \nabla \oplus \nabla$ adapted to the family D of Dirac operators and the connection $\mathbb{A} = A + \nabla$ which is adopted to family A. We denote by $\mathbb{B}_t$ and $\mathbb{A}_t$ their rescaled versions. The rescaled curvature $\mathbb{F}_t := \mathbb{B}_t^2$ has the form $t\mathsf{D}^2 + \mathbb{F}_{t[+]}$ where $\mathbb{F}_{t[+]}$ is the following differential operator with differential form coefficients of positive degree

$$(3.1) \qquad \mathbb{F}_{t[+]} =: \begin{pmatrix} \nabla^2 & t^{1/2}(D\nabla + \nabla D) \\ t^{1/2}(D\nabla + \nabla D) & \nabla^2 \end{pmatrix}.$$

We shall be dealing with the structure of heat operator of rescaled curvature , so we explain briefly how to construct it. Let $e^{-t\mathsf{D}^2} = e^{-tD^*D} \oplus e^{-tDD^*}$ be the heat operator of $(\mathsf{D}, P^\epsilon \oplus P^{\bar\epsilon})$ and let R be a family of smoothing operator. The heat kernel of perturbed family $\mathsf{D}_s := \mathsf{D} + sR$, for $0 \leq s \leq 1$, is given by Voltera formula:

$$e^{-t\mathsf{D}_s} = e^{-t\mathsf{D}^2} + \sum_{k=1}^{\infty} (-t)^k I_k(t\mathsf{D}_s, sR),$$

$$I_k(t\mathsf{D}, sR) := \int_{\triangle_k} e^{-s_0 t\mathsf{D}^2} sR \, e^{-s_1 t\mathsf{D}} \, sR \ldots e^{-s_{k-1} t\mathsf{D}^2} \, sR \, e^{-s_k t\mathsf{D}^2},$$

$$\triangle_k = \{(s_0, s_1, \ldots, s_k) \in \mathbb{R}^{k+1} | s_i \geq 0 \; ; \; \sum_i s_i = 1.\}$$

Because R is smoothing, the operator $e^{-t\mathsf{D}}R$ has a smooth kernel for $t \geq 0$ and $\|e^{-t\mathsf{D}}R\|_\ell \leq C(\ell)\|R\|_\ell$. So

$$\|I_k(t\mathsf{D}, R)\|_\ell \leq \frac{C(\ell)^{k+1}\|R\|_\ell^k}{k!},$$

which implies the convergence of above sum in C^ℓ-norm. But it is clear from above Voltera formula that

(3.2) $$e^{-t\mathsf{D}_s} - e^{-t\mathsf{D}} = o(1), \quad \text{at } t = 0,$$

so the smoothing perturbation sR has no effect on asymptotic behavior of heat operator at $t = 0$. Let $\mathbb{B}_s := \mathbb{B} + sR$ be the superconnection adopted to perturbed family D_s. Its rescaled curvature has the form $\mathbb{F}_{s,t} = t\mathsf{D}_s + \mathbb{F}_{s,t[+]}$, where

(3.3) $$\mathbb{F}_{s,t[+]} = \mathbb{F}_{t[+]} + O(t).$$

Heat operator of supercurvature $\mathbb{F}_{s,t}$ is given again by Voltera formula

(3.4) $$e^{-\mathbb{F}_{s,t}} = e^{-t\mathsf{D}_s^2} + \sum_{k=1}^{\dim B} (-1)^k I_k(t\mathsf{D}s, \mathbb{F}_{s,t[+]}),$$

It should be clear from this construction that $e^{-\mathbb{F}_{s,t}}$ is a vertical family of smoothing operator with coefficients in $\Lambda^*(B)$. So its supertrace is finite and defines an element in $\Omega^*(B)$. Although $I_k(t\mathsf{D}_s, \mathbb{F}_{s,t[+]})$ depends on involved operator in a rather complicated way, its asymptotic behavior at $t = 0$ is simple to describe. At first it follows from (3.2) and (3.3) that $I_k(t\mathsf{D}_s, \mathbb{F}_{s,t[+]}) = I_k(t\mathsf{D}, \mathbb{F}_{t[+]}) + o(1)$. Moreover, in expression $I_k(t\mathsf{D}, \mathbb{F}_{t[+]})$ the contribution of off-diagonal operators in (3.1) can be neglected, at $t = 0$, because these operators are multiplied by $t^{1/2}$. So $I_k(\mathsf{D}, \mathbb{F}_{t[+]}) = I_k(\mathsf{D}, \nabla^2 \oplus \nabla^2) + o(1)$ and we get the following relation that we will use later. This relation holds in $\Omega^*(B)$ with respect to C^ℓ-norms

$$e^{-\mathbb{F}_{s,t}} = e^{-t\mathsf{D}^2 + \nabla^2 \oplus \nabla^2} + o(1).$$

The Chern character of the rescaled perturbed superconnection is defined by $\mathrm{Ch}(\mathbb{B}_{s,t}) := \mathrm{STr}\, e^{-\mathbb{F}_{s,t}}$. To investigate the relation between $\mathrm{Ch}(\mathbb{B}_{s,t})$ and $\mathrm{Ch}(\mathbb{B}_t)$ we recall a general theorem about superconnections an super bundles, cf. [BGV, Theorem 9.17]. Let B_τ be a smooth family of superconnection on a differential superbundle. The following formula holds [BGV, Theorem 9.17])

(3.5) $$\frac{d}{d\tau}\mathrm{Ch}(\mathsf{B}_\tau) = -\mathrm{d}\,\mathrm{STr}\,(\frac{\mathrm{d}\,\mathsf{B}_\tau}{d\tau} \mathrm{e}^{-\mathsf{B}_\tau^2}) \in \Omega^*(\mathsf{B}).$$

If we apply this formula to s-dependent family of superconnections $\mathbb{B}_{s,t}$ we get

$$\mathrm{Ch}(\mathbb{B}_{1,t}) - \mathrm{Ch}(\mathbb{B}_t) = -\mathrm{d}\int_0^1 \mathrm{STr}(\frac{\mathrm{d}\,\mathbb{B}_{s,t}}{\mathrm{ds}} \mathrm{e}^{-\mathbb{F}_{s,t}})\,\mathrm{ds}$$

Therefore the perturbation R does not affect the class of Chern character in de-Rham cohomology of B. Impact of this perturbation on behavior at $t = \infty$ may be crucial. In fact there is a general methods (see [AS, Lemma 2.1]) to construct a self adjoint perturbation R such that $\dim \ker(\mathsf{D}+R)$ be independent of $b \in B$. In this case $\ker(\mathsf{D}+R)$ is a smooth finite dimensional super vector bundle over B, so it determines a class in $K^0(B)$. This class, being independent of the perturbation, is denoted by $[\mathrm{ind}\,\mathsf{D}]$ and is called the analytical index of family D. We summarize these discussion in following proposition

Proposition 5. *We can assume that* $\dim \ker(\mathsf{D}_b)$ *is independent of* $b \in B$ *by perturbing by self adjoint smoothing operators. This perturbation does not affect the class of Chern form* $\mathrm{Ch}(\mathbb{B}_t)$ *in* $H^*_{dr}(B)$ *not its behavior when t goes toward* 0. *Moreover one has*

$$(3.6) \qquad e^{-\mathbb{F}_t} = e^{-t\mathsf{D}^2 + \nabla^2 \oplus \nabla^2} + o(1).$$

With this assumption we have $[\mathrm{ind}\,\mathsf{D}] = [\ker \mathsf{D}] \in \mathrm{K}^0(\mathrm{B})$.

Our proof of theorem 4 is based on a precise study of the behavior of $\mathrm{Ch}(\mathbb{B}_t)$ at $t = 0$ and $t = \infty$ and comparing them. Following above proposition, $\ker \mathsf{D}$ is a vector bundle on B and the formal difference of its even and odd parts i.e, $\ker(D, P^\epsilon) - \ker(D^*, P^{\bar{\epsilon}})$ represents the index class $[\mathrm{ind}\,\mathsf{D}] \in \mathrm{K}^0(\mathrm{B})$. Let Q_0 be the projection on $\ker \mathsf{D}$ which is a smooth family of vertical smoothing operators. It is clear that $\nabla_0 = Q_0 \nabla Q_0$ is a connection on the vector bundle $\ker \mathsf{D}$. Therefore the differential form $\mathrm{Str}\, e^{-\nabla_0^2}$ is closed and provides a representation of $\mathrm{Ch}(\mathrm{ind}\,\mathsf{D}) \in \mathrm{H}^*_{\mathrm{Dr}}(\mathrm{B})$.

Proposition 6.   (1) *The following convergence occurs in* $\Omega^*(M)$ *with respect to uniform* C^ℓ*-norm for each* $\ell \in \mathbb{N}$

$$\lim_{t \to +\infty} \mathrm{Ch}(\mathbb{B}_t) = \mathrm{Ch}(\ker \mathsf{D}, \nabla_0).$$

    (2) *For* $t > 0$, *the Chern form* $\mathrm{Ch}(\mathbb{B}_t)$ *is closed and its class in de-Rham cohomology* $H^*_{dr}(B)$ *is independent of t, in particular*

$$\mathrm{Ch}(\mathbb{B}_t) = \mathrm{Ch}(\nabla_0) \in \mathrm{H}^*_{\mathrm{dr}}(\mathrm{B}) \quad \textit{for} \ \ t > 0.$$

Proof Consider the following orthogonal decomposition of superbundles

$$(3.7) \qquad \mathcal{E} \oplus \mathcal{E} = \ker \mathsf{D} \oplus \mathrm{Im}\,\mathsf{D}.$$

Heat operator $e^{-t\mathsf{D}}$ *is a family of smoothing non negative operators parameterized by the compact set* B. *So there is a uniform gap around* 0, *in the spectrum of each element of this family. This simple observation and general properties of graded nilpotent algebras can be used to get the following relation with respect to above direct sum decomposition. (see* [BGV, page 290] *)*

$$(3.8) \qquad e^{-\mathbb{B}_t} \sim \begin{pmatrix} e^{-\nabla_0^2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & O(t^{-1}) \end{pmatrix}.$$

Notice that the convergence with respect to C^ℓ*-norm is implicit in above asymptotic formula which proves the first part of the proposition. To prove the second part we apply relation (3.5) to family* $\mathbb{B}_t$. *So for* $t_2, t_1 > 0$

$$\mathrm{Ch}(\mathbb{B}_{t_2}) - \mathrm{Ch}(\mathbb{B}_{t_1}) = -\mathrm{d} \int_{t_1}^{t_2} \mathrm{STr}\,(\frac{\mathrm{d}\,\mathbb{B}_t}{\mathrm{d}t} e^{-\mathbb{F}_t})\,\mathrm{d}t.$$

If we regard this relation in de-Rham cohomology group $H^*_{dr}(B)$ *we obtain the desired relation in the second part of the proposition.* $\qquad\square$

Remark 6. *If* t_1 *goes to* ∞ *in above relation then one can deduce the following stranger relation*

$$(3.9) \qquad \mathrm{Ch}(\mathbb{B}_{t_2}) - \mathrm{Ch}(\nabla_0) = \mathrm{d} \int_{t_2}^{\infty} \mathrm{STr}\,(\frac{\mathrm{d}\,\mathbb{B}_t}{\mathrm{d}t} e^{-\mathbb{F}_t})\,\mathrm{d}t \in \Omega^*(\mathrm{B}),$$

provided the integral in the right hand side is finite. To prove the convergence of this integral, we notice that

$$\frac{d\,\mathbb{B}_t}{dt} = \frac{1}{2t^{1/2}} \begin{pmatrix} 0 & 0 \\ \mathsf{D} & 0 \end{pmatrix},$$

with respect to decomposition (3.7). This relation with (3.8) give rise to

$$\mathrm{STr}\,(\frac{\mathrm{d}\,\mathbb{B}_t}{\mathrm{dt}} e^{-\mathbb{F}_t}) = \mathrm{O}(\mathrm{t}^{-3/2}).$$

So the above integral is convergent at $t = \infty$.

Now we shall study the behavior of $\mathrm{Ch}(\mathbb{B}_t)$ when $t \to 0$. For this purpose we give, as in the previous section, an another description of heat operator $e^{-\mathbb{F}_t}$ in term of heat operator on double of F and heat operator on half cylinder fibration $\mathbb{R}^{\geq 0} \times F'$ with typical fiber $\mathbb{R}^{\geq 0} \times \partial M$. As before, let π denote the projection on second summand. All local structures in this cylindrical case are exactly the same of the collar neighborhood U, so, for example, the Dirac operator takes the form (2.5).

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3.1. Index density of local conditions on family of half cylinders.

Let $\mathcal{E}_0 = \mathcal{E}_0^+ \oplus \mathcal{E}_0^-$ be the bundle over B whit typical fiber $C^\infty(\partial M_b, S^+ \oplus S^-)$. Let $\nabla_0 = \nabla_0^+ \oplus \nabla_0^-$ be a connection on $\mathcal{E}_0$. Then $\bar{\nabla} = \pi^* \nabla_0$ is a connection on $\mathcal{E} := \pi^* \mathcal{E}_0$ which preserves the grading. We denote by $\bar{\mathbb{B}}$ the superconnection $D + \bar{\nabla} \oplus \bar{\nabla}$ acting on $\mathcal{E} \oplus \mathcal{E}$. We consider $\mathbb{R}^{\geq 0} \times N$ with local boundary condition P^+. Here N may denote each one of N_i and for simplicity we drop the index i. From relation (3.6); as far as we are interested in the asymptotic behavior of heat operators at $t = 0$; we can replace $\bar{\mathbb{F}}_t := \bar{\mathbb{B}}_t^2$ by fallowing operator

$$(3.10) \qquad \begin{pmatrix} -t\partial_u^2 & 0 \\ 0 & -t\partial_u^2 \end{pmatrix} + \pi^* \begin{pmatrix} tA^2 + \nabla_0^2 & 0 \\ 0 & tA^2 + \nabla_0^2 \end{pmatrix}$$

Relation (3.6) can be applied to superconnection $\mathbb{A} = A + \nabla_0$ where, $\nabla_0 = \pi^* \nabla_0^+ \oplus \pi^* \nabla_0^-$. This provides $e^{-\mathbb{A}_t^2} = e^{-(tA^2 + \nabla_0^2)} + o(1)$, so

$$(3.11) \qquad e^{-\bar{\mathbb{F}}_t} = e^{-\bar{\mathsf{F}}_t} + o(1),$$

where

$$\bar{\mathsf{F}}_t^2 := \begin{pmatrix} -t\partial_u^2 & 0 \\ 0 & -t\partial_u^2 \end{pmatrix} + \pi^* \begin{pmatrix} \mathbb{A}_t^2 & 0 \\ 0 & \mathbb{A}_t^2 \end{pmatrix}.$$

Notice that the operator ∂_u commutes with all other operators involved in above expression, so the results of previous section can be used to give explicit expression for $e^{-\bar{\mathsf{F}}_t}$. For example its even part, acting on $\phi^+(u,y) \oplus \phi^-(u,y))$ subjected to boundary conditions (2.6) and (2.7) is

$$\frac{1}{\sqrt{4\pi t}} \begin{pmatrix} \exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t}) & 0 \\ 0 & \exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t}) \end{pmatrix} \otimes e^{-\mathbb{A}_t^2}$$

A similar formula, by exchanging the diagonal coefficients, gives the odd part of the heat kernel of $\bar{\mathsf{F}}_t$ with boundary conditions (2.8). This expression and relation (3.11) provide together an asymptotic explicit formula for $\bar{K}_+(t,u,v,y,z)$, the heat kernel of $\bar{\mathbb{F}}_t$. So we get the following formula, with respect to C^ℓ-norm, for the supertrace density of $e^{-\bar{\mathbb{F}}_t}$ as a function of t and u

$$\bar{K}_+(t,u) = \int_N \operatorname{str} \bar{\mathsf{K}}_+(\mathsf{t},\mathsf{u},\mathsf{u},\mathsf{y},\mathsf{y}) \, \mathrm{d}\mathsf{y}$$

$$= -\frac{\operatorname{STr} e^{-\mathbb{A}_t^2}}{\sqrt{\pi t}} e^{-\frac{u^2}{t}} + o(1) \in \Omega^*(B).$$

Notice that in above formula, the integration is performed in linear space $\Omega^*(B)$ so it makes sens. The case $\epsilon = -$ produces the same expression with the opposite sign. So by intergrating on $[0, 1/2]$, with respect to u, we get the following asymptotic equality at $t = 0$

$$\bar{K}_{\epsilon_i}(t) = \int_0^{1/2} \bar{K}_{\epsilon_i}(t,u) \, du \sim -\epsilon_i \frac{1}{2} \operatorname{STr} e^{-\mathbb{A}_t^2} + o(1) \in \Omega^*(\mathrm{B}).$$

In other word, the following relation holds in $\Omega^*(B)$ with C^ℓ-topology

$$(3.12) \qquad K_{\epsilon_i}(t) = -\epsilon_i \operatorname{Ch}(\mathbb{A}_{\mathrm{it}}) + o(1).$$

3.2. Index density of APS conditions on family of half cylinders.

Now consider the family of half cylinder $\mathbb{R}^{\geq 0} \times N \to B$ with APS boundary condition. *We assume that* $\dim \ker(A_b)$ *does not depend on* $b \in B$. Here A_b indicates the Dirac operator on N_b. So vector spaces $\ker A_b$, put together, form the vector bundle $\ker A$ on B. We denote the lifting of this bundle, via π, by $\mathcal{E}_0$ which is again a vector bundle over B. In the same way and with notation of proposition 2, infinite dimensional vector spaces $E^\pm(b)$ together form an infinite dimensional bundles over B. These bundles can be lifted to bundles $\mathcal{E}^\pm$ over B. These bundles are isomorphism via unitary operator $\mathcal{U}$. Let $\bar{\nabla}^\pm$ be connections on sections of $\mathcal{E}^\pm$ such that $\mathcal{U}\bar{\nabla}^\pm \mathcal{U}^{-1} = \bar{\nabla}^\mp$. These connections are assumed be constant along $\mathbb{R}^{\geq 0}$, i.e. they are lifting of connections by π. We assume also a connection $\bar{\nabla}_0$ on $\mathcal{E}_0 = \pi^* \ker A$. Put $\bar{\nabla} = \bar{\nabla}^- \oplus \bar{\nabla}_0 \oplus \bar{\nabla}^+$ and consider the superconnection $\tilde{\mathbb{B}} = D + \bar{\nabla} \oplus \bar{\nabla}$. In view of relation (3.6), heat operator of the rescaled supercurvature $\tilde{\mathbb{F}}_t$, up to a term of order $o(1)$, is equal to the heat operator of

$$(3.13) \qquad \mathsf{F}_t := \begin{pmatrix} -tD^*D & 0 \\ 0 & -tDD^* \end{pmatrix} + \begin{pmatrix} \bar{\nabla}^2 & 0 \\ 0 & \bar{\nabla}^2 \end{pmatrix}$$

We shall prove that the supertrace of the heat operator $e^{-\mathbb{F}_t}$, when it is restricted to $\mathcal{E}^+ \oplus \mathcal{E}^-$, vanishes when t goes toward 0. For this purpose, using the Voltera formula, we have

$$(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^+} = \sum_k (-1)^k I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2)$$

where

$$I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2) := \int_{\triangle_k} e^{-s_0 tD^*D} \bar{\nabla}_+^2 e^{-s_1 tD^*D} \bar{\nabla}_+^2 \dots e^{-s_{k-1}tD^*D} \bar{\nabla}_+^2 e^{-s_k tD^*D}.$$

In other hand, from the first part of proposition 2,

$$(e^{-tDD^*})_{|\mathcal{E}^-} = \mathcal{U}(e^{-tD^*D})_{\mathcal{E}^+} \mathcal{U}^{-1}.$$

This relation and $\bar{\nabla}_- = \mathcal{U}\bar{\nabla}_+ \mathcal{U}^{-1}$ imply

$$I_k(tD^*D_{|\mathcal{E}^-}, \bar{\nabla}_-^2) = \mathcal{U}I_k(tD^*D_{|\mathcal{E}^+}, \bar{\nabla}_+^2)\mathcal{U}^{-1},$$

so

$$(e^{-tDD^*+\bar{\nabla}^2})_{|\mathcal{E}^-} = \mathcal{U}(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^+} \mathcal{U}^{-1}.$$

A similar discussion, using again the proposition 2, implies the following relation in $\Omega^*(B)$ with C^ℓ-topology

$$(e^{-tDD^*+\bar{\nabla}^2})_{|\mathcal{E}^+} = \mathcal{U}(e^{-tD^*D+\bar{\nabla}^2})_{|\mathcal{E}^-} \mathcal{U}^{-1} + o(1) \quad \text{at} \ \ t = 0.$$

Therefore

$$\text{STr} \, e^{-\bar{\mathbb{F}}_t} = \text{Tr} \, (e^{-t\partial_u^2 + \bar{\nabla}_0^2})_{|E_0} - \text{Tr} \, (e^{-t\partial_u^2 + \bar{\nabla}_0^2})_{|E_0} + o(1).$$

Since ∂_u commutes with $\bar{\nabla}_0^2$, relation (2.21) can be used to get

$$\text{STr}_N \, e^{-\mathbb{F}_t}(t, u) = -\frac{e^{\frac{-u^2}{t}}}{\sqrt{\pi t}} \text{str} \, (e^{-\bar{\nabla}_0^2})_{|\ker A}.$$

Summarizing these discussion, we get the following relation in $\Omega^*(B)$ with C^ℓ-topology

(3.14)
$$\int_0^{\frac{1}{2}} \text{STr}_N \, e^{-\bar{\mathbb{F}}_t}(t, u) \, du = -\frac{1}{2}\text{Ch}(\ker A, \nabla_0) + o(1).$$

3.3. **Index theorem for families of Dirac operators.** We recall that all differential operators and geometric structures we have on F are of product form in collar neighborhood U, so they can be extended, smoothly, to double fibration $F \sqcup_{F'} F$. In opposite direction, let $\tilde{\nabla}$ be a connection on $\mathcal{E} \to F \sqcup_{F'} F$ and let $\bar{\nabla}_i$ be the connections on half-cylinder fibrations discussed in previous subsections. Then

(3.15)
$$\nabla := \sum_i f_1 \bar{\nabla}_i g_1 + f_2 \tilde{\nabla} g_2,$$

defines a connection on $\mathcal{E} \to B$. Here functions f_1, f_2, g_1 and g_2 are defined in subsection 2.3. Let $\tilde{\mathbb{F}}_t$ be the rescaled supercurvature on double $F \sqcup_{F'} F$ and let $\tilde{K}_t$ be the kernel of associated heat operator. We denote by $\bar{K}_{t\epsilon_i}$ and $K_{t\epsilon_i}$ the fundamental solutions of $e^{-\bar{\mathbb{F}}_t}$ and $e^{-\mathbb{F}_t}$ with respect to boundary condition P^{ϵ_i}. At first we prove the following asymptotic formula at $t = 0$

(3.16)
$$K_{t\epsilon_i} = \sum_i f_1 \bar{K}_{t\epsilon_i} g_1 + f_2 \tilde{K}_t g_2 + o(1).$$

For this purpose we use relation (3.6) and Voltera formula to deduce

$$e^{-\mathbb{F}_t} = e^{-t\mathsf{D}^2} + \sum_{k=1}^{\dim B} (-1)^k I_k(t\mathsf{D}, \nabla \oplus \nabla) + o(1),$$

$$I_k(t\mathsf{D}, \nabla \oplus \nabla) := \int_{\triangle_k} e^{-s_0 t\mathsf{D}^2} (\nabla \oplus \nabla) e^{-s_1 t\mathsf{D}^2} (\nabla \oplus \nabla) \dots e^{-s_{k-1}t\mathsf{D}^2} (\nabla \oplus \nabla) e^{-s_k t\mathsf{D}^2}.$$

In this formula the boundary conditions are implicit in smooth family of heat operators $e^{-t\mathsf{D}^2}$. Following relation is the family version of (2.24) which is clearly true

$$e^{-t\mathsf{D}^2} = f_1 e^{-t\bar{\mathsf{D}}^2} g_1 + f_2 e^{-t\tilde{\mathsf{D}}^2} g_2 + o(1).$$

In other hand $\nabla = \sum_i f_1 \bar{\nabla}^i g_1 + f_2 \tilde{\nabla} g_2$, since $f_1 g_1 = g_1$ and $f_2 g_2 = g_2$, we get

$$I_k(t\mathsf{D}, \nabla \oplus \nabla) = \sum_{\epsilon_i = \pm} f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, f_1(\bar{\nabla}^i \oplus \bar{\nabla}^i) g_1) g_1 + f_2 I_k(\tilde{\mathsf{D}}, f_2(\tilde{\nabla} \oplus \tilde{\nabla}) g_2) g_2$$

$$+ \text{ finite sum of operators of form } M(t) h\, e^{-st\bar{\mathsf{D}}^2}\, k N(t) + o(1)$$

In last line of above expression, $M(t)$ and $N(t)$ are vertical smoothing operators with differential form coefficients such that, far from diagonal, their kernels goes exponentially toward zero when t goes to 0, while h and k are smooth functions on F such that $supp(hk) \subseteq [\frac{1}{4}, 1] \times F' \subset U$. So using relations (2.15) and (2.23) we deduce $M(t) h\, e^{-st\bar{\mathsf{D}}^2}\, k N(t) = o(1)$ when $t \to 0$. Using again the relation (2.15) and (2.23) we have

$$g_1 e^{-t\bar{\mathsf{D}}^2} g_1 = e^{-t\bar{\mathsf{D}}^2} g_1 + o(1) \quad \text{and} \quad f_1 e^{-t\bar{\mathsf{D}}^2} f_1 = f_1 e^{-t\bar{\mathsf{D}}^2} + o(1).$$

Therefore

$$f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, f_1(\bar{\nabla}^i \oplus \bar{\nabla}^i) g_1) g_1 = f_1 I_k(\bar{\mathsf{D}}_{\epsilon_i}, \bar{\nabla}^i \oplus \bar{\nabla}^i) g_1 + o(1)$$

which complete the proof of relation (3.16) by considering (3.6).

Now we take the supertrace of relation (3.16). Clearly the contribution of interior term is zero. Contribution of boundary terms, $\bar{K}_{t\epsilon_i} g_1$ on $[1/2, 1]$ go to 0 when t goes toward 0 (see relation (2.15) and (2.23)). Since $f_1 = g_1 = 1$ on $[0, 1/2]$, using relations (3.12) and (3.14) we obtain the following asymptotic formula at $t = 0$

(3.17) $$\mathrm{STr}\,(\mathrm{e}^{-\mathbb{F}^\epsilon_\mathrm{t}}) = -\frac{1}{2} \sum_{\epsilon_\mathrm{i} = \pm} \epsilon_\mathrm{i}\, \mathrm{Ch}(\mathbb{A}_\mathrm{it}) - \frac{1}{2} \sum_{\epsilon_\mathrm{i} = 0} \mathrm{Ch}(\ker \mathrm{A}_\mathrm{i}, \nabla_0) + o(1) \in \Omega^*(\mathrm{B})$$

All involved differential forms in above expressions are closed, so we can regard this relation in $H^*_{dr}(B)$. According to proposition 6 and its analogue for superconnection $\mathbb{A}$, the class of Chern forms do not depend on parameter t. So the image of term $o(1)$ in $H^*_{dr}(B)$ vanishes and we obtain the following equality

$$\mathrm{Ch}[\mathrm{ind}(\mathrm{D}, \mathrm{P}^\epsilon)] = -\frac{1}{2} \sum_{\epsilon_\mathrm{i} = +} \mathrm{Ch}[\mathrm{ind}\mathrm{A}_\mathrm{i}] + \frac{1}{2} \sum_{\epsilon = -} \mathrm{Ch}[\mathrm{ind}\mathrm{A}_\mathrm{i}] - \frac{1}{2} \sum_{\epsilon_\mathrm{i} = 0} \mathrm{Ch}(\ker \mathrm{A}_\mathrm{i}) \in \mathrm{H}^*_{\mathrm{dr}}(\mathrm{B}).$$

This complete the proof of the theorem 4.

Remark 7. *Above proof is based on assumption that* $\dim \ker A_b$ *is independent of* $b \in B$. *This assumption is satisfied for some interesting cases, eg. for family of signature operators or, when fibers have a metric with positive scalar curvature, for standard Dirac operator twisted by flat vector bundles. However this assumption may be removed by considering smooth perturbations of boundary operators, or putting more general spectral boundary condition by means of spectral projections introduced in* [MP]. *In the later case the standard tool for analysing the heat kernel will be the Melrose's b-calculus for family.*

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